Teleportation of entangled states and dense coding using a multiparticle quantum channel

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Abstract

A set of protocols for teleportation and dense coding tasks with the use of a N particle quantum channel, presented by entangled states of the GHZ class, is introduced, when $N > 2$. Using a found representation for the multiparticle entangled states of the GHZ class, it has shown, that for dense coding schemes enhancement of the classical capacity of the channel due from entanglement is $N/N-1$. If $N > 2$ there is no one-to-one correspondence between teleportation and dense coding schemes in comparison with the EPR channel is exploited. A set of schemes, for which two additional operations as entanglement and disentanglement are permitted, is considered.

1 Introduction

The large number of the quantum information tasks are based on a channel represented by entangled states. An EPR pair or a two particle quantum channel is a main resource for dense coding and teleportation to be attractive now for many applications. Dense coding has been introduced by Bennett et al [1] and demonstrated in the optical experiment with polarized photons by Zeilinger et al [2] and for continuous variables by Peng et al [3]. The quantum teleportation protocol proposed by Bennett et al [4] has been implemented by several groups [5].

A lot of teleportation and dense coding schemes, including its applications has been considered by many authors both for discrete and continuous variables. Recently an attempt to classify all schemes has been made by Werner [6]. Generally, this problem is very difficult and the main results have been obtained in the case of so-called *tight* schemes realized with minimal resources. It has been found a one-to-one correspondence between all *tight* schemes of teleportation and dense coding. It is worth noting that an EPR pair as a two-particle quantum channel is used in the *tight* schemes. The obtained result is significant for practice because it tells, that if one can teleport a qubit, then he can perform dense coding using the same experimental arrangement without any additional resources.

In this paper a quantum channel presented by a multiparticle entangled state of the GHZ class is considered for teleportation and dense coding. When the channel involves more then two particles, its features becomes quite complicated and all schemes are not *tight*. For particular case, one finds the GHZ channel based on the triplet of the GHZ form. If we wish to exploit the GHZ channel for teleportation, for example, then the task can't be accomplished by a generalization of the usual protocol simply. Considering how to transmit an unknown qubit by the GHZ channel, Karlsson et al have shown, that the unknown state can be recovered by one of the two receivers, but not both [7]. What kind of a two-qubit state can be teleported through the GHZ channel? This problem has been considered by Marinatto et al [8]. It has been found that the general two-qubit state can't be transmitted perfectly but a pure entangled states can. This conclusion is in agreement with result obtained in Ref. [9]. With the use of the GHZ channel, the conditional teleportation of two entangled qubits has developed [10].

The main goal of this work is to consider the multiparticle quantum channel for informational tasks as teleportation and dense coding. In a particular case

of the GHZ channel a scheme for distributing a mixed qubit state with two parties is discussed. The set of the questions, we study in this paper, is the following: what is the dense coding schemes, whether we could have an enhancement of the channel capacity, whether the teleportation resources can be used directly for dense coding similarly *tight* schemes, what kind of teleportation and dense coding schemes can be created using certain additional resources such as entanglement and disentanglement operations.

The paper is organized as follows. First, we discuss the main resources and consider *tight* schemes, then teleportation and dense coding protocols are introduced for the GHZ channel and a telecloning scheme is presented. In the next section using the found representation of N particle entangled states we establish some main features of the multiparticle channel and calculate its capacity. Then a collection of teleportation and dense coding schemes is briefly discussed when such operations as entanglement and disentanglement are permitted.

2 Tight schemes

Following to Werner [6], we consider a set of objects to create some teleportation and dense coding schemes. The set includes an observable F, a collection of unitary operators T, an entangled state ω to be a quantum channel. Let the Hilbert spaces of the involved systems have the same dimension d, and ω is the N particle entangled state. Two parameters d and N play the key role. If $N = 2$, one can find schemes called *tight*.

Let the observable F be a complete set of the N-particle states, $\sum_x F_x = 1$, $F_x = |\Phi_x(N)\rangle\langle\Phi_x(N)|$, where x is one of the d^N elements of an output parameter space $X(d^N)$. In general these pure states can be not maximally entangled. We assume, that T is the collection of the m - particle unitary operators $U_x(m)$, completely positive, that transform input state of the channel ω to output state $U_x(m)\omega U_x^\dagger(m)$. Let the N - particle quantum channel $\omega = |\Omega\rangle\langle\Omega|$ be shared N parties $A, B, C, ..,$ spatially separated, where Ω can be one of the states of $\Phi_x(N)$. Then all resources are

$$R = \{\omega, x \in X(d^N), \Phi_x(N), U_x(m)\} \tag{1}$$

Using (1) the teleportation and dense coding schemes can be obtained. We consider only the qubit case, for which $d = 2$. Note, the multiparticle quantum channel has new properties due from operators $U_x(m)$. When $m \geq 2$, these operators may be non local and the main resources (1) is not a set of the LOCC in contrast the *tight* schemes.

If $N = 2$, one finds a two-particle quantum channel, represented by the EPR state, say of the form $\Omega = (|00\rangle + |11\rangle)/\sqrt{2}$. The channel is shared two parties, Alice and Bob. Here the observable F is described by the Bell states $\Phi_x(2) = \Phi^+, \Psi^+, \Phi^-, \Psi^-$ and the set of unitary operators consists of the Pauli and the identity operators $U_x(1) = \mathbb{1}, \sigma_z, \sigma_x, -i\sigma_y$, where $\sigma_y = i\sigma_x\sigma_z$. In this case the space X has four elements $x = 0, 1, 2, 3$ by which the 2 bits of information can be encoded. The following map is possible

$$x \leftrightarrow \Phi_x(2) \leftrightarrow U_x(1) \tag{2}$$

or in more details

$$\begin{array}{cccc} x & \tilde{x} & \Phi_x(2) & U_x(1) \\ 0 & 00 & \Phi^+ & \mathbb{1} \\ 1 & 01 & \Psi^+ & \sigma_z \\ 2 & 10 & \Phi^- & \sigma_x \\ 3 & 11 & \Psi^- & -i\sigma_y \end{array} \tag{3}$$

where $\tilde{x}$ is encoded by binary notation of x.

Reading (2) as $\Phi_x(2) \mapsto x \mapsto U_x(1)$ one finds teleportation, that allows Alice sending to Bob an unknown qubit

$$\zeta = \alpha|0\rangle + \beta|1\rangle \tag{4}$$

where $|\alpha|^2 + |\beta|^2 = 1$. In accordance with the teleportation protocols, Alice performs the Bell state measurement on her half of ERP pair and the unknown qubit. Outcomes of the measurement x can be encoded with the use of the unitary operators $U_x(1)$, by which Bob acts on his half of EPR pair to recover the unknown state. One ERP pair and the 2 bits of classical information are needed for teleporting a single qubit.

Reading (4) as $x \mapsto U_x(1) \mapsto \Phi_x(2) \mapsto x$, one can find the dense coding scheme, that permits Alice

sending of a two-bit message to Bob, manipulating one bit only. Because of the 2 bits of information $\tilde{x} = 00, 01, 10, 11$ can be encoded by the four operators $U_x(1)$, Alice can generate the Bell basis acting on her particle of EPR pair

$$\Phi_x(2) = (\mathbb{1} \otimes U_x(1))\Omega \tag{5}$$

Then the 2 bits of information are storied in four orthogonal states, that can be distinguished, if Bob performs the Bell state measurement. The properties of this channel can be described by the Holevo bound, that tells us that the classical capacity of this quantum channel increases twice because of entanglement.

The considered schemes of teleportation, which is perfect, and dense coding are called *tight* [6] in the sense of the required resources. These resources are the EPR channel, the 2 bits of information, the Bell state measurement and a collection of the one-particle unitary operators. Werner has shown that for all tight schemes there is a one-to-one correspondence between teleportation and dense coding.

3 The GHZ channel

If three particle entanglement is used instead of EPR pair one finds a channel which features are more complicated. This channel shared multi users, Alice, Bob and Claire, allows not only transmitting of quantum and classical information by teleportation and dense coding, but distributing quantum states between several parties by coping or telecloning.

There is a complete set of the three particle entangled states of the form

$$\begin{aligned} &(|000\rangle \pm |111\rangle)/\sqrt{2}, \quad (|001\rangle \pm |110\rangle)/\sqrt{2} \\ &(|010\rangle \pm |101\rangle)/\sqrt{2}, \quad (|011\rangle \pm |100\rangle)/\sqrt{2} \end{aligned} \tag{6}$$

Without loss of generality a triplet of the GHZ form can be chosen as the quantum channel, whose three particles A, B and C are shared Alice, Bob and Claire. Then $\Omega = |GHZ\rangle$, where

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{ABC} \tag{7}$$

All schemes based on the GHZ channel, are not tight and the theorem by Werner can't guarantee a one-to-one correspondence between teleportation and dense coding schemes.

3.1 Teleportation

One of the main features of the GHZ channel is perfect transmitting of the two particle entangled states of the EPR form

$$\zeta = \alpha|01\rangle + \beta|10\rangle \tag{8}$$

where $|\alpha|^2+|\beta|^2 = 1$. It has been shown by Marinatto and Weber, that the general state of a pair of qubits cannot be transmitted through the GHZ channel[8].

If Alice wishes to send the entangled state ζ of the qubit 1 and 2, she needs to perform the measurement on particles 1, 2 and her particle A of the GHZ channel. The task can be accomplished, if the observable $\Phi_x(3)$ has the form of product of a superposition state of particle 1, say $\pi^\pm = (|0\rangle \pm |1\rangle)/\sqrt{2}$, and the Bell states of particles 2 and A. This basis is described by states in which two particles only are maximally entangled, it has the form

$$\Phi_x(3) = \{\pi_1^\pm \otimes \Phi_{2A}^\pm, \pi_1^\pm \otimes \Psi_{2A}^\pm\} \tag{9}$$

The measurement given by (9) projects particles B and C into the state that is connected by unitary transformation with the unknown state

$$|BC\rangle_x = U_x(2)|\zeta\rangle \tag{10}$$

where $|BC\rangle_x = \langle\Phi_x|\zeta\rangle \otimes |GHZ\rangle/Prob(x)$, and probability of all outcomes $Prob(x) = 1/8$.

The unitary operators can be chosen in the factorized form

$$U_x(2) = B_x \otimes C_x \tag{11}$$

where B_x and C_x acts on the Bob and Claire particle. It means that Alice cannot rotate the qubit of Bob and vice versa, but transformations are correlated because of they have the same indexes. It is a case of LOCC. In more details the presented protocol

reads

$$
\begin{array}{cccccc}
x & \Phi_x(3) & |BC\rangle_x & B_x & C_x \\
0 & \pi^+\otimes\Phi^+ & \beta|00\rangle+\alpha|11\rangle & \sigma_x & 1 \\
1 & \pi^+\otimes\Phi^- & \beta|00\rangle-\alpha|11\rangle & i\sigma_y & 1 \\
2 & \pi^-\otimes\Phi^+ & -\beta|00\rangle+\alpha|11\rangle & -i\sigma_y & 1 \\
3 & \pi^-\otimes\Phi^- & -\beta|00\rangle-\alpha|11\rangle & -\sigma_x & 1 \\
4 & \pi^+\otimes\Psi^+ & \beta|11\rangle+\alpha|00\rangle & 1 & \sigma_x \\
5 & \pi^+\otimes\Psi^- & \beta|11\rangle-\alpha|00\rangle & 1 & -i\sigma_y \\
6 & \pi^-\otimes\Psi^+ & -\beta|11\rangle+\alpha|00\rangle & 1 & i\sigma_y \\
7 & \pi^-\otimes\Psi^- & -\beta|11\rangle-\alpha|00\rangle & 1 & -\sigma_x
\end{array}
\qquad (12)
$$

Two points may be made about it. First, in (12) there are only four vectors $|BC\rangle_x$ which describe four different physical states of a system. So that, outcomes $x = 0$ and $x = 3$, result in the states $\beta|00\rangle + \alpha|11\rangle$ and $-(\beta|00\rangle + \alpha|11\rangle)$ to be equal up to a phase factor, that has no physics reason. Indeed, they can be obtained by different way, for example $\beta|00\rangle_{BC} + \alpha|11\rangle_{BC} = (\sigma_x \otimes 1\!\!1)|\zeta\rangle_{BC} = (i\sigma_y \otimes \sigma_z)|\zeta\rangle_{BC}$. The second is more important. The simple observation shows that the task can be accomplished, if operators $U_x(2)$ are not factorized. For the considered outcomes, say $x = 0$, one finds $(\sigma_x \otimes 1\!\!1)C_{BC}C_{CB}C_{BC}|\zeta\rangle_{BC} = \beta|00\rangle_{BC} + \alpha|11\rangle_{BC}$, where C_{ct} is CNOT operation, c is a control bit, t is a target bit, $c, t = B, C$. This case results in the non-local operations.

Teleportation of the considered two particle entangled state can be achieved by the usual protocol, that recommends to do it one-by-one. It needs two EPR pairs instead of the GHZ channel, which seems to be less expensive in comparison with two entangled pairs.

3.2 Dense coding

Is it possible to use the above teleportation resources given by (12) for dense coding similar the case of the EPR channel? The answer is not, but a scheme of dense coding can be achieved.

Let a sender wishes to transmit a three bit message. The 3 bits of information $000, 001, \ldots, 111$ can be encoded by a set of the eight states D_x each of which is obtained from the GHZ state using a collection of the unitary operators $U_x(2)$ in accordance with (5), for example. Let the two particle operators be chosen factorized in the form (11), then equation (5) reads

$$|D_x\rangle_{ABC} = 1\!\!1 \otimes B_x \otimes C_x|GHZ\rangle_{ABC} \qquad (13)$$

An appropriate collection of the Pauli operators permits the sender to generate the complete set of states D_x, given by (6). All these states are well distinguishable by measurement, which outcomes encode the three bit message. Then one finds a dense coding scheme, that is described by the map of the form

$$
\begin{array}{cccc}
\tilde{x} & B_x & C_x & D_x = B_xC_x|GHZ\rangle_{ABC} \\
000 & 1\!\!1 & 1\!\!1 & |000\rangle+|111\rangle \\
001 & 1\!\!1 & \sigma_x & |001\rangle+|110\rangle \\
010 & \sigma_z & 1\!\!1 & |000\rangle-|111\rangle \\
011 & \sigma_z & \sigma_x & |001\rangle-|110\rangle \\
100 & \sigma_x & 1\!\!1 & |010\rangle+|101\rangle \\
101 & \sigma_x & \sigma_x & |011\rangle+|100\rangle \\
110 & -i\sigma_y & 1\!\!1 & |010\rangle-|101\rangle \\
111 & -i\sigma_y & \sigma_x & |011\rangle-|100\rangle
\end{array}
\qquad (14)
$$

In (14) we have omitted the normalization factor $1/\sqrt{2}$ in D_x.

To find the capacity of this GHZ channel it needs to calculate the Holevo function

$$C(\{p_i\}, \rho) = S(\rho) - \sum_i p_i S(\rho_i) \qquad (15)$$

where $\rho = \sum_i p_i\rho_i$, ρ_i are the density matrices of the states sent to receiver according to probabilities p_i and $S(\rho)$ is the von Neumann entropy. For the considered case $\rho_i = |D_i\rangle\langle D_i|$ and the channel is represented by the maximally entangled state, then assuming $p_i = 1/8$, one finds $C = S(\sum_i |D_i\rangle\langle D_i|/8) = 3$, hence per transmitted bit $C/2 = 3/2$, that is the classical capacity of the quantum channel. It means, the channel capacity due from entanglement increases in 3/2 times. However, this result is clear without any calculations. Because of the presented protocol allows sending the 3 bits of information manipulating two bits only, then profit is 3/2, which is enhancement of the capacity. Also it is clear, that it results from the entanglement, which degree has to be maximum.

Inspection of (12) shows that the teleportation resources are inapplicable for dense coding. The reason is that the states D_x obtained in accordance with equation (13), where operators C_x, B_x are given by (12), is not a complete set. Also, being suitable for dense coding the complete set given by (6) can't be used for teleportation because of outcomes of measurement depend on the unknown state. Therefore there is not a one-to one correspondence between these schemes as for *tight* ones. However a connection can be established. Indeed, two sets D_x, given by (11), and $\Phi_x(3)$ denoted by (12) can be transformed from one to another by the unitary operation, say of the form $\Phi_x = H_B C_{BC} D_x$, where H_B is the Hadamard transformation of particle B. It follows from (13), that eight distinguishable states can be obtained by manipulating only two bits of the GHZ channel as follows

$$\Phi_x(3) = H_B C_{BC}(\mathbb{1} \otimes B_x \otimes C_x)|GHZ\rangle \qquad (16)$$

where B_x, C_x are given by (14).

The equation (16) tells, that the measurement from the teleportation scheme may be used for dense coding. For that it needs to replace the operations $B_x \otimes C_x \rightarrow H_B C_{BC}(B_x \otimes C_x)$ before sending the message. Then the 3 bits of information are stored in the complete set of states to be well distinguished by the projective measurement of Φ_x. Note, the unitary operations become non local, what is the one of the particular qualities of the three-particle channel.

Indeed, for dense coding schemes the GHZ channel can be created by operations $U_x(2)$. Let only two qubits A and B of the three ones A, B and C be entangled, in other words the EPR channel and the ancilla qubit C are introduced, then the GHZ state can be prepared by the way $C_{BC}(|\Phi^+\rangle_{AB} \otimes |0\rangle_C) = |GHZ\rangle_{ABC}$. This transformation can be inserted into each unitary operator $U_x(2)$, that becomes more complicated because of $U_x(2) \rightarrow U_x(2)C_{BC}$. In the same time it looks as the EPR pair is used instead of the GHZ channel.

3.3 Telecloning

The GHZ channel shared three parties A,B and C, spatially separated, allows distributing information with B and C. Two copies of an unknown state can be produced by a teleportation protocol so that we shall call it telecloning. We consider a scheme, whose main resources are the Bell state measurement and set of the Pauli operators as for *tight* schemes.

Let using the above resources Alice wishes to sent to Bob and Claire an unknown qubit in a mixed state

$$\rho_1 = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1| \qquad (17)$$

Then combined state is $\rho_1 \otimes |GHZ\rangle\langle GHZ|$. After the Bell state measurement on the unknown qubit 1 and the qubit A from the GHZ channel the reduced density matrix of particles B and C has the form $\rho_{BC} = \lambda_0|bb\rangle\langle bb| \pm \lambda_1|\bar{b}\bar{b}\rangle\langle\bar{b}\bar{b}|$, where $b = 0, 1$, $\bar{b} = 1 - b$. Two bits of information allows Bob and Claire perform the local unitary operations to obtain the state

$$\rho'_{BC} = \lambda_0|00\rangle\langle 00| + \lambda_1|11\rangle\langle 11| \qquad (18)$$

One find ρ'_{BC} to be a separable and classically correlated state. In the same time both receivers have in their hands the unknown state, since the reduced matrices read $\rho_B = \rho_C = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|$.

As result, two perfect copies of an unknown mixed state can be made by teleporting. Note, these copies are not independent, that follows from the non cloning theorem. However each receiver can manipulate his state independently, if and only if he performs local unitary operations. It is not true, when one of them decides to make a measurement of his state.

4 The N-particle quantum channel

Some main features of the teleportation and dense coding schemes can be summarized, considering a multi particle channel, for which the following mapping plays the key role

$$x \leftrightarrow U_x(m) \leftrightarrow \Phi_x(N) \qquad (19)$$

By contrast the *tight* schemes, it seems to be a hard problem to proof it generally, therefore we will restrict several facts.

4.1 Representation for multiparticle states of the GHZ class

Considering resources given by (1) one finds the factor m in operator $U_x(m)$ to be important, as it might be noticed from the GHZ channel. For teleportation schemes m is a number of particles on which an unknown state is transmitted, in other words, m shows how many particles can be teleported by the channel. For dense coding m indicates a number of particles for manipulating to send the N bit message and ratio N/m becomes the classical capacity of the quantum channel due from entanglement.

A one-to-one correspondence $x \leftrightarrow \Phi_x(N)$, where x is one of the 2^N outcomes of the von Neumann measurement, described by a complete but not over complete set of projectors, is clear. By contrast the map $U_x(m) \leftrightarrow \Phi_x(N)$ is not so trivial. Similarly (5) one can write

$$\Phi_x(N) = (\mathbb{1}^{\otimes(N-m)} \otimes U_x(m))\Omega \qquad (20)$$

where $\mathbb{1}^{\otimes(N-m)}$ is tensor product of $N-m$ identity operators $\mathbb{1}\otimes\mathbb{1}$.... According to the following rough dimension count, factor m can be established from (20). In fact, being the N qubit state, vector $\Phi_x(N)$ has the 2^N components. Any the m qubit operator $U_x(m)$ has the 2^{2m} matrix elements. Then for correspondence between $\Phi_x(N)$ and $U_x(m)$, it needs

$$m \geq \frac{N}{2} \qquad (21)$$

Indeed, these reasons are true not only in the qubit case, but for arbitrary dimension of the Hilbert space d.

A simple observation allows to obtain the factor m with more accuracy. Let the channel be represented by a maximally entangled state Ω of the GHZ class

$$|\Omega\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}) \qquad (22)$$

where $|b\rangle^{\otimes N}$ is tensor product $|b\rangle \otimes \ldots |b\rangle$, that is a state of the N independent qubits in the Hilbert space $\mathcal{H}_1 \otimes \ldots \mathcal{H}_N$, $b = 0, 1$.

Preposition. *The set of the N particle vectors*

$$\Phi_{b_1 b_2 \ldots b_N}(N) = \qquad (23)$$

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |b_2 \ldots b_N\rangle + (-1)^{b_1}|1\rangle \otimes |\bar{b}_2 \ldots \bar{b}_N\rangle)$$

from the Hilbert space $\mathcal{H}_1 \otimes \ldots \mathcal{H}_N$, *where* $b_1 = 0, 1$ $|b_2 \ldots b_N\rangle = |b_2\rangle \otimes \ldots |b_N\rangle$, *and* $|b_k\rangle = 0, 1$, $\bar{b}_k = 1 - b_k$ *is the orthonormal basis in* $\mathcal{H}_k$ *for each* $k = 2, \ldots N$, *is the complete set of maximally entangled states.*

If $N = 2$, one finds the Bell states $\Phi_{b_1 b_2}(2) = (|0\, b_2\rangle + (-1)^{b_1}|1\, \bar{b}_2\rangle)$, that are generated by two classical bits $b_1, b_2 = 0, 1$. When $N = 3$ three bits $b_k = 0, 1$, $k = 1, 2, 3$ generate the set $D_x(3) = \Phi_{b_1 b_2 b_3}(3)$, given by (6). Also $\Omega = \Phi_{00\ldots 0}(N)$ belongs to the collection (23).

Proof. Each of the states, that has the form (23), is maximally entangled in the sense of the reduced von Neumann entropy $E = S(\rho(1))$, where $\rho(1)$ is the one particle density matrix. It follows from (23), that for any particle $\rho(1) = \mathbb{1}/2$, then $E = 1$, and entanglement is maximum. Also one finds the considered set of states to be complete, because of condition

$$\sum_{b_1 \ldots b_N = 0,1} |\Phi_{b_1 b_2 \ldots b_N}(N)\rangle\langle\Phi_{b_1 b_2 \ldots b_N}(N)| = 1 \qquad (24)$$

that directly results from the completeness of the collections $|b_k\rangle$.

Note, all possible entangled states can't be written in the form (23). It represents the GHZ like class only and, for example, W - states introduced by Cirac et al [11] and ZSA (Zero Sum Amplitude) - states proposed by Pati [12], that can't be transformed from the GHZ states by local operations, have another form.

The next observation plays the key role. Equation (23) tells, that to generate all states of the set, it needs manipulating $N-1$ qubits of any fixed state from this collection. In other words, there is a set of operators $U_x(m)$, including identity operator, for which

$$m = N - 1 \qquad (25)$$

It is in agreement with (21). It results in equation (23) takes the form of (20)

$$\Phi_{b_1 b_2 \ldots b_N}(N) = (\mathbb{1} \otimes U_{b_1 b_2 \ldots b_N}(N-1))\Omega \qquad (26)$$

where the string of bits $b_1 b_2 \ldots b_N$ is binary notation of x, $x = 0, \ldots 2^N - 1$.

Generally the question of existence and uniqueness of operators $U_{b_1b_2\dots b_N}(N-1)$ seems to be rather hard problem and we shall discuss simple examples. Let all operators be factorized and have the form of product of the one particle operators

$$U_{b_1b_2\dots b_N}(N-1) = U_{b_1b_2}(1)\otimes U_{b_3}(1)\dots U_{b_N}(1) \quad (27)$$

Assume each of the transformations $U_{b_1b_2}(1), U_{b_3}(1)\dots$ can be represented by the Pauli operators. If $N=2$ one finds $U_{b_1b_2}(1) = \sigma_x^{b_2}\sigma_z^{b_1}$, and in accordance with (23) and (26)

$$(\mathbb{1}\otimes\sigma_x^{b_2}\sigma_z^{b_1})(|00\rangle + |11\rangle) = |0b_2\rangle + (-1)^{b_1}|1\bar{b}_2\rangle \quad (28)$$

where $b_1, b_2 = 0, 1$. When $N > 2$, the choice $U_{b_k}(1) = \sigma_x^{b_k}$ for $k = 3,\dots N$ is suitable

$$(\mathbb{1}\otimes\sigma_x^{b_2}\sigma_z^{b_1}\otimes\sigma_x^{b_3}\otimes\dots\sigma_x^{b_N})\Omega$$
$$= \frac{1}{\sqrt{2}}(|0b_2\dots b_N\rangle + (-1)^{b_1}|1\rangle\otimes|\bar{b}_2\dots\bar{b}_N\rangle) \quad (29)$$

The obtained equations (26) and (29) tell that the complete set of the N qubit entangled states of the GHZ class can be associated with a set of the $N-1$ qubit operators, that generate all these states from one of them. In other words the mapping given by (19) can be justified. Indeed, the choice of operators may be not unique. For example, if $N=3$, there is a case for which it is possible to manipulate one qubit instead of two qubits

$$(\mathbb{1}\otimes\sigma_z\otimes\sigma_x)(|001\rangle - |110\rangle)$$
$$= (\mathbb{1}\otimes\mathbb{1}\otimes i\sigma_y)(|001\rangle - |110\rangle) \quad (30)$$

The representation given by (26) is not true for any states to be separable, it needs entanglement not less then two particles. A state of N's independent qubits can be write in the form $\Phi_{b_1b_2\dots b_N}(N) = |b_1\dots b_N\rangle$. When bits take their value 0 and 1, the obtained set is complete, but it is important, that it can be generated from one of them by manipulating all qubits. Then instead of (26), one finds $\Phi_x(N) = U_x(N)\Omega$. If two qubits are maximally entangled and others are independent, then such state has the form $\Phi_{b_1b_2}(2)\otimes|b_3\dots b_N\rangle$, where $\Phi_{b_1b_2}(2)$ is one of the Bell states. Entanglement allows to obtained complete set manipulating $N-1$ qubits of an initial state, say, $\Omega' = \Phi_{00}\otimes|0\dots0\rangle$. It is important, that Ω' does not belong to the GHZ class by contrast Ω, given by (22). From the physical point of view it is clear, that both states Ω' and Ω can't be transformed from one to another by local operations. For example, if $N=3$ one finds transformation

$$(\mathbb{1}\otimes C_{23})\Phi_{00}(2)\otimes|0\rangle = |GHZ\rangle \quad (31)$$

where $\Phi_{00}(2) = (|00\rangle+|11\rangle)/\sqrt{2}$. Here the CNOT operation C_{23} involves two qubits simultaneously, that is an interaction between two systems, that results in entanglement. When the GHZ is prepared, as initial state Ω, the complete set can be obtained in accordance with (26), but operators takes the nonlocal form $U_{b_1b_2b_3} = (\mathbb{1}\otimes U_{b_1b_2}(1)\otimes U_{b_3}(1))(\mathbb{1}\otimes C_{23})$. This example indicate the fact, that a complete set of the N qubit entangled states can be generated performing the non local operations on $N-1$ particles.

4.2 Capacity of the channel

Using (29), one finds a dense coding scheme, that allows sending a N-bit message by manipulating $N-1$ bits. To discuss capacity of the channel due from entanglement it needs to replace $\Omega \to \alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N}$, where $|\alpha|^2+|\beta|^2 = 1$. Now the channel is not assumed to be maximally entangled and its measure of entanglement, given by the reduced von Neumann entropy, has the form

$$E = -|\alpha|^2\log|\alpha|^2 - |\beta|^2\log|\beta|^2 \quad (32)$$

The Holevo function reads $C(\{p_x\},\rho) = S(\rho/2^N)$, where all probabilities are equal and $p_x = 1/2^N$. For the considered channel

$$\rho = \sum_x(\mathbb{1}\otimes U_x(N-1))|\Omega\rangle\langle\Omega|(\mathbb{1}\otimes U_x(N-1))^\dagger$$
$$= \sum_{b_1\dots b_N=0,1}|\Phi'_{b_1b_2\dots b_N}(N)\rangle\langle\Phi'_{b_1b_2\dots b_N}(N)| \quad (33)$$

Let operators $U_x(N-1)$ be factorized and have the form (29), then

$$\Phi'_{b_1b_2\dots b_N}(N) = \alpha|0\rangle\otimes|b_2\dots b_N\rangle \quad (34)$$
$$+(-1)^{b_1}\beta|1\rangle\otimes|\bar{b}_2\dots\bar{b}_N\rangle$$

All these states are generated from Ω by the local unitary transformations, then their degree of entanglement is E, given by (51).

If $\alpha = \beta = 1/\sqrt{2}$, then $E = 1$ and the channel is maximally entangled. In the same time it implies the important fact, that the set of states becomes complete in accordance with (24) and all these states can be well distinguishable by measuring. For a maximally entangled channel the equation (33)is the condition of completeness and density matrix ρ takes the form $\rho = \rho(1)^{\otimes N}$, where the single particle density matrix is $\rho(1) = \mathbb{1}/2$.

When the channel can be not maximally entangled, one finds

$$\rho = \rho'(1) \otimes \rho(1)^{\otimes (N-1)} \tag{35}$$

where $\rho'(1) = |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$ is the one particle density operator. Before calculating the classical capacity of the channel, given by the Holevo function, note that it can be normalized per transmitted bit. For the considered protocol there are $N-1$ bits, that Alice transmits to Bob. Then using (35), capacity of the channel has the form

$$c = \frac{C(\{p_x\}, \rho)}{N-1} = 1 + \frac{E}{N-1} \tag{36}$$

It takes maximum $c_{max} = N/N-1$, when $E = 1$. It means that entanglement results in increasing of the classical capacity of the N particle channel by $N/N-1$ times. Indeed, this result is clear without calculating. If a channel permits sending of N-bits of classical information manipulating $N-1$ qubits, then profit is $N/N-1$, that is enhancement of the channel capacity per transmitted bits.

4.3 Sufficient tight and other schemes

The main resources, given by (1), are sufficient also for teleportation of the entangled states, that have the form $\zeta = \alpha|0\rangle^{\otimes(N-1)} + \beta|1\rangle^{\otimes(N-1)}$. The task can be accomplished by the N particle channel Ω and the $N/N-1$ bits of classical information per transmitted particle, due from a N particle measurement. The measurement involves all particles to be teleported and one particle from Ω. It can be described by observable of the form (9), where $\pi^{\pm} \to (\pi^{\pm})^{\otimes(N-2)}$ [9].

The presented teleportation and the dense coding schemes are based on the mentioned resources to be sufficient and minimal for these tasks. This set of schemes we shall name *sufficient tight* schemes by contrast the other ones, that can be obtained if some additional resources are permitted.

Let introduce the k bit operators $En(k)$ and $Den(k)$ to be transformations of entanglement and disentanglement. Their existence is the open question generally, but in a particular case one finds $CNOT$ and the Hadamard gates to be useful. We assume, that these operators $En(k)$ and $Den(k)$ transform any state of k independent qubits into entangled state and vice versa. Some modifications of schemes arise when these operations are permitted.

It is well known, that operator $Ent(2)$, say of the form $Ent(2) = C_{12}$, plays the key role in the one-bit teleportation, when an unknown qubit is entangled with ancilla [13]. It results in one bit of classical information is needed for sending the qubit. Indeed, the one bit protocol can be directly generalized for teleportation of two entangled qubits, for which two bits of information are required.

For dense coding schemes all modifications reduce to preparing of the channel state Ω and revising of observable just as the way the considered GHZ channel. Suppose, there is a collection of N qubits, in which the k particles are independent and the remainder $N-k$ qubits are entangled. Let only one qubit from entanglement be in the receiver hand. For preparing Ω it needs entanglement of all particles, that can be achieved with the use of operator $En(k+1)$. It looks as all operators $U_x(N-1)$ from a *sufficient tight* scheme are replaced as follows $U_x \to U_x \otimes Ent(k+1)$. Revising of observable or measurement is another independent step. Assume, the N bit message is already encoded by entangled states $\Phi_x(N)$. Then before measuring, these states can disentangled by operator $Den(n)$, that produces n independent qubits, where $n \leq N$. The measurement becomes more simple because observable can be described by a set of states in which not all qubits, or maybe all of them are independent. The cost of modification is $U_x \to Den(n) \otimes U_x$. As result, the main revising of the dense coding *sufficient tight* schemes

is

$$U_x \to Den(n) \otimes U_x \otimes Ent(k+1) \qquad (37)$$

In the case of the GHZ channel (37) takes the form $U_x \to H_B C_{BC} \otimes U_x(2) \otimes C_{BC}$.

Both the entanglement and the disentanglement operators are useful for modification of the *sufficient tight* teleportation schemes for which there are some ways how to prepare the channel state Ω. As the N - particle channel allows transmitting perfectly only the $N-1$ particle entangled state of the form $\zeta(N-1) = \alpha|0\rangle^{\otimes(N-1)} + \beta|1\rangle^{\otimes(N-1)}$ then one of the general idea of modification is disentanglement of the state to be teleported: $\zeta(N-1) \to (\alpha|0\rangle + \beta|1\rangle) \otimes |N-2\rangle$, where $N-2$ particles in state $|N-2\rangle$ can be entangled with two ancilla qubits, say in the EPR state, for preparing the quantum channel Ω. Combining with disentanglement operations it results in a collection of schemes, which based on one EPR pair and the Bell state measurement as one of the initial resource, however the measurement will involve not all particles to be teleported.

We illustrate the generalization of the one-bit teleportation protocol, considering for simplicity how to transmit two entangled qubits. The task can be accomplished, if an unknown state $\zeta = (\alpha|00\rangle + \beta|11\rangle)_{12}$ is entangled with an EPR pair of the form $\Omega = (|00\rangle + |11\rangle)_{AB}/\sqrt{2}$ as follows $C_{A2}|\zeta\rangle_{12} \otimes |\Omega\rangle_{AB}$. Then the joint measurement of the qubit 1 and 2 in basis $\pi_1^{\pm} \otimes |b\rangle_2$, $b = 0, 1$ projects the remainder qubits A and B onto the state to be equal to the unknown state up to unitary transformations. Note, here the non Bell state measurement allows teleporting two entangled qubits by the 2 bits of classical information, however it is not the LOCC protocol.

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